The Complex Propagation Constant γ

Recall that the current and voltage along a transmission line have the form:

$$V(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z}$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{+\gamma z}$$

where Z_0 and γ are **complex constants** that describe the properties of a transmission line. Since γ is complex, we can consider both its **real** and **imaginary** components.

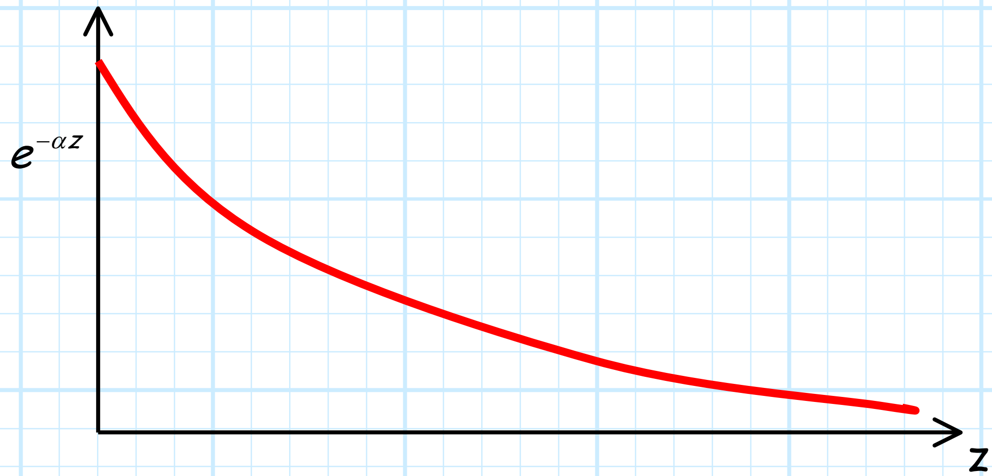
$$\begin{aligned} \gamma &= \sqrt{(R + j\omega L)(G + j\omega C)} \\ &\doteq \alpha + j\beta \end{aligned}$$

where $\alpha = Re\{\gamma\}$ and $\beta = Im\{\gamma\}$. Therefore, we can write:

$$e^{-\gamma z} = e^{-(\alpha + j\beta) z} = e^{-\alpha z} e^{-j\beta z}$$

Since $\left|e^{-j\beta z}\right| = 1$, then $e^{-\alpha z}$ alone determines the **magnitude** of $e^{-\gamma z}$.

I.E., $\left|e^{-\gamma z}\right| = e^{-\alpha z}$.



Therefore, α expresses the **attenuation** of the signal due to the loss in the transmission line.

Since $e^{-\alpha z}$ is a real function, it expresses the **magnitude** of $e^{-\gamma z}$ only. The **relative phase** $\phi(z)$ of $e^{-\gamma z}$ is therefore determined by $e^{-j\beta z} = e^{-j\phi(z)}$ only (recall $\left|e^{-j\beta z}\right| = 1$).

From Euler's equation:

$$e^{j\phi(z)} = e^{j\beta z} = \cos(\beta z) + j\sin(\beta z)$$

Therefore, βz represents the **relative phase** $\phi(z)$ of the oscillating signal, as a function of transmission line position z. Since phase $\phi(z)$ is expressed in radians, and z is distance (in meters), the value β must have units of :

$$\beta = \frac{\phi}{z} \qquad \frac{\text{radians}}{\text{meter}}$$

The **wavelength** λ of the signal is the distance $\Delta z_{2\pi}$ over which the relative phase changes by 2π radians. So:

$$2\pi = \phi(z + \Delta z_{2\pi}) - \phi(z) = \beta\, \Delta z_{2\pi} = \beta\lambda$$

or, rearranging:

$$\boxed{\beta = \frac{2\pi}{\lambda}}$$

Since the signal is oscillating in **time** at rate ω rad/sec, the **propagation velocity** of the wave is:

$$v_p = \frac{\omega}{\beta} = \frac{\omega\lambda}{2\pi} = f\lambda \quad \left(\frac{m}{sec} = \frac{rad}{sec}\,\frac{m}{rad}\right)$$

where f is **frequency** in cycles/sec.

Recall we originally considered the transmission line current and voltage as a function of time and position (i.e., $v(z,t)$ and $i(z,t)$). We assumed the time function was sinusoidal, oscillating with frequency ω:

$$v(z,t) = \mathbf{Re}\left\{V(z)e^{j\omega t}\right\}$$

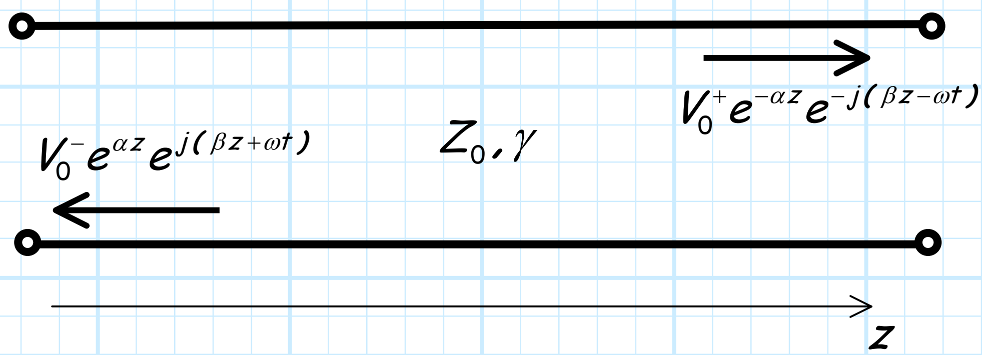
$$i(z,t) = \mathbf{Re}\left\{I(z)e^{j\omega t}\right\}$$

Now that we **know** $V(z)$ and $I(z)$, we can write the original functions as:

$$v(z,t) = Re\left\{V_0^+ e^{-\alpha z} e^{-j(\beta z - \omega t)} + V_0^- e^{\alpha z} e^{j(\beta z + \omega t)}\right\}$$

$$i(z,t) = Re\left\{\frac{V_0^+}{Z_0} e^{-\alpha z} e^{-j(\beta z - \omega t)} - \frac{V_0^+}{Z_0} e^{\alpha z} e^{j(\beta z + \omega t)}\right\}$$

The first term in each equation describes a wave **propagating** in the ***+z*** direction, while the second describes a wave propagating in the **opposite** (-z) direction.



Each wave has **wavelength**:

$$\lambda = \frac{2\pi}{\beta}$$

And **velocity**:

$$v_p = \frac{\omega}{\beta}$$